A Quartic Polynomial Based Approach for Multi-Sinusoidal Frequency Estimation

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***Abstract*—This article presents a quartic polynomial based approach for frequency estimation of multiple real sinusoids in additive white Gaussian noise (AWGN). The estimation of the peak and ensemble of subsequent frequencies in the spectral data is achieved using a simple slope-based approach. The performance of the frequency estimation is studied by computing the estimation error for varying signal-to-noise ratio. The performance is also studied for varying values of signal length and frequency deviation. Simulation studies are presented to establish the validity of the estimation algorithm for multiple sinusoids. The proposed approach is fast and efficient, rendering it suitable for application to real-time implementations.**

***Index Terms*—multiple real sinusoids, frequency estimation, quartic polynomial, Discrete Fourier Transform (DFT) peaks**

I. Introduction

The problem of frequency estimation, although a classic research problem, still garners significant interest in the research community owing to its relevance in several engineering disciplines [1–3]. Although a plethora of research literature is available in the field of frequency estimation, most of the works address frequency estimation for complex sinusoids. Applications of these methods to the parameter estimation of real sinusoids is challenging as it will result in high estimation bias and interference between the positive and negative frequency components [4]. Also, seldom do the approaches address real-time applications. A desirable requirement for an algorithm to be applied for real-time applications is to ensure that the computational complexity is less and/or the algorithm is fast. To address these requirements, the quartic polynomial based frequency estimation technique was devised by the authors. The quartic polynomial frequency estimator [5] is a fast and computationally efficient estimator since the estimate is obtained from the roots of the fourth-degree polynomial, which already has known analytic solutions.

In this article, the authors extend the method developed in [5] to perform the frequency estimation for a multi-sinusoidal input signal in Additive White Gaussian Noise (AWGN). This consideration is more in line with practical scenarios as, often, the incoming signals will be a composite sinusoid consisting of multiple frequencies. The composite signal used consists of varying amplitude values, varying normalized frequency values raging between 0.1 and 0.4 and phase values uniformly distributed between 0 and 2π. The spectrum of the composite signal is evaluated, and all the peaks are identified. The peaks are then sorted in the descending order of its magnitude starting from the dominant peak and the estimate of the peak is computed. The estimated peak is removed, and the next dominant peak is identified. This is continued till all the peaks of the spectrum are obtained. Further, the quartic polynomial based frequency estimation for multiple sinusoids is implemented for the estimation of each of the frequencies. An approach based on the analytic expressions whose solutions are already known will only require a straightforward implementation of its solution, thus making the implementation fast and computationally less intensive. This makes the method non-iterative and highly relevant for implementation in real-time applications.

The rest of the paper is organized as follows. Section II discusses relevant literature in multi-sinusoidal frequency estimation. Section III describes the quartic polynomial approach and its extension to multi-frequency estimation. Section IV presents the results of the simulation studies validating the proposed approach, followed by the conclusion presented in Section V.

II. Related Works

Frequency estimation algorithms available in the literature are broadly classified into time or frequency domain approaches, iterative or non-iterative approaches and for real or complex sinusoidal signal. Dutra *et al*. [4] implemented a spectrum matching approach for frequency estimation to mitigate the ill-effects of superposition of positive and negative frequency components in real sinusoids. The quartic polynomial frequency estimator [5] was a fast and computationally efficient estimator since the estimate was obtained from the roots of the fourth-degree polynomial, which has a known analytical solution. Thus, this approach is non-iterative and highly relevant for implementation in real-time applications. Frequency estimation was performed for real harmonic sinusoids using the linear prediction approach [6]. So *et al*. [7] implemented an extension of [6] in which two algorithms, based on constrained weighted least squares approach, were developed for frequency estimation of multiple real sinusoids. Frequency

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estimation in real sinusoids for the case of non-uniform sampling was explored in [8]. A subspace-based approach was adopted for multiple real sinusoids reducing the computational load in comparison with the ESPRIT algorithm [9]. The parameter estimation was performed in [10] for biased multi-sinusoidal signals using an adaptive approach.

Using a DFT-based approach, in [11], the spectral leakage was alleviated by generating the analytic signal corresponding to each spectral component of the real sinusoid. Serbes [12] achieved the frequency estimation of parameters by performing interpolation on the shifted DFT coefficients, with a dependency on the DFT shift, number of iterations and frequency separation of the peaks. Two parametric approaches in the frequency domain were proposed for estimation of real and complex sinusoids in [13]. An iterative approach was adopted by interpolating the Fourier coefficients of the weighted data samples [14]. Transforming a non-linear problem into linear regression, Vediakova *et al*. [15] performed the finite time estimation of frequencies of a real multi-sinusoidal signal. Djukanović and Popović-Bugarin [16] performed the model order approximation and frequency estimation for complex and real sinusoids, where the peak detection was done using Neymann-Pearson criterion and frequency estimation was carried out using three-point spectral maximization.

Liu *et al*. [17] proposed a method for frequency estimation by locating the maximum DFT peak and two Discrete time Fourier Transform (DTFT) peaks on the same side of the maximum DFT peak. This three-point periodogram approach was computed for the case of complex sinusoids. Izacard *et al*. [18] extended the approach dealing with a neural network model for frequency estimation in which the local maxima was observed at the peak frequency position. They also added a neural network module to determine the model order. Sajedian and Rho [19] proposed a deep learning-based framework for determining the frequency of noisy sinusoidal signal. They implemented a three-layer neural network for a Signal Noise Ratio (SNR) of 25 dB. Dreifuerst and Heath [20] proposed a neural network architecture for model order estimation as well as for the estimation of signal parameters, accounting for the losses encountered due to quantization as a modelling effect. An internal reconstruction of the signal was incorporated to enhance learning and a worst-case threshold was introduced for analyzing the efficacy of their algorithm. Almayyali and Hussain [21] presented a deep learning-based approach for single frequency estimation and showed that very few layers, as low as two layers, was sufficient for accurate frequency estimation explained in the context of sensor communications and Software Defined Radio. Katyara *et al*. [22] presented a fuzzy logic-based method for the estimation and classification of the harmonics of a sinusoidal signal validated by evaluating the loss function and accuracy. The methods described in [18–22] are expected to be computationally intensive as the approach adopted was based on neural networks/machine learning.

III. The Quartic Polynomial Approach for Estimation of Multiple Sinusoids

In this section, the quartic polynomial based frequency estimation approach is briefly discussed for the estimation of multiple real sinusoids. The number of sinusoids in the composite signal, referred to as the model order, is represented by M. The signal length or the number of samples in a signal is represented as N. The term f_s represents the sampling frequency. The sampling interval, t_n, is given as $t_n = n\Delta$ and $\Delta = 1/f_s$. The flowchart as shown in Fig. 1 represents the system model for multi-sinusoidal frequency estimation.

The signal model for a combination of M real sinusoids with distinct frequency values ω_i, where $\omega_i = 2\pi f_i$, is given as:

$$s_i[n] = \sum_{i=0}^{M-1} a_i \cos(\omega_i t_n + \phi_i) + y_n \tag{1}$$

where n represents the index taking on discrete integer values between [0, N–1]. The term a_i represents the amplitude of each sinusoid and ϕ_i represents the phase parameter uniformly distributed between [0, 2π].

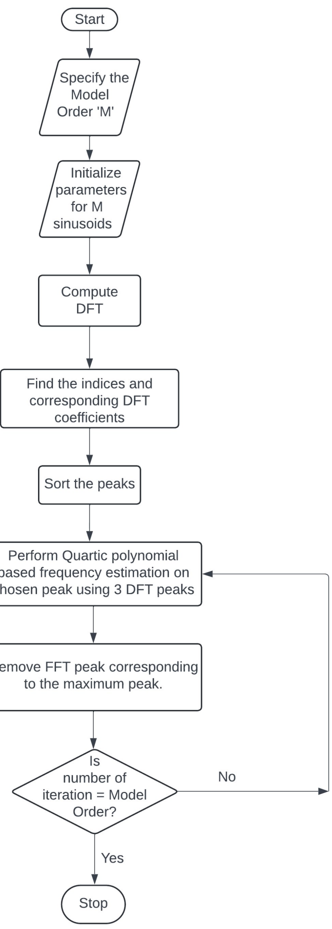

Fig. 1. Flowchart representing the system model for multi-sinusoidal frequency estimation.

In the expression for the noisy real sinusoid of Eq. (1), the term y_n represents the Additive White Gaussian Noise (AWGN). The sinusoids are chosen such that they are separated by at least one DFT bin. This is because frequency domain methods are not capable of resolving peaks that are spaced at less than $1/N$ [23] in terms of cycles per sampling interval.

A. Derivation of Polynomial Coefficients

The complex FFT coefficients, $F_k(\omega)$, corresponding to the Fourier transform of each sinusoid are computed as given in Eq. (2):

$$F_k(\omega) = \sum_{n=0}^{N-1} s_k[n] e^{-j\omega t_n} = \frac{a}{2}\sum_{m=1}^{2} T_{m_k}(\omega) e^{-j\phi_c(2m-3)} \quad (2)$$

where the terms $T_{m_k}(\omega)$ represent the positive and negative frequency terms corresponding to each frequency of the multi-sinusoidal signal. The value of k ranges from 1 to M, corresponding to the presence of M real sinusoids. The index m takes the values 1 or 2, representing the positive or negative frequency term respectively of the considered kth sinusoid.

Using the terms $T_{1_k}(\omega)$ and $T_{2_k}(\omega)$, the time domain expressions for the three adjacent spectral components centered around the peak to be estimated are given as:

$$T_{1_k}\left(\omega_k - \frac{\omega_s}{N}\right) = \frac{\sin(\delta\Delta N/2)}{\sin(\delta\Delta/2 + \pi/N)} \frac{e^{j(\frac{\delta\Delta N}{2})}}{e^{j(\frac{\delta\Delta}{2}+\frac{\pi}{N})}} \quad (3)$$

$$T_{1_k}(\omega_k) = \frac{\sin(\delta\Delta N/2)}{\sin(\delta\Delta/2)} \frac{e^{j(\frac{\delta\Delta N}{2})}}{e^{j(\frac{\delta\Delta}{2})}} \quad (4)$$

$$T_{1_k}\left(\omega_k + \frac{\omega_s}{N}\right) = \frac{\sin(\delta\Delta N/2)}{\sin(\delta\Delta/2 - \pi/N)} \frac{e^{j(\frac{\delta\Delta N}{2})}}{e^{j(\frac{\delta\Delta}{2}-\frac{\pi}{N})}} \quad (5)$$

$$T_{2_k}\left(\omega_k - \frac{\omega_s}{N}\right) = \frac{\sin\left[(2\omega_k+\delta)\frac{\Delta N}{2}\right] e^{-j[(2\omega_k+\delta)\frac{\Delta N}{2}]}}{\sin\left[(2\omega_k+\delta)\frac{\Delta}{2} - \frac{\pi}{N}\right] e^{-j[(2\omega_k+\delta)\frac{\Delta}{2}-\frac{\pi}{N}]}} \quad (6)$$

$$T_{2_k}(\omega_k) = \frac{\sin\left[(2\omega_k+\delta)\frac{\Delta N}{2}\right] e^{-j[(2\omega_k+\delta)\frac{\Delta N}{2}]}}{\sin\left[(2\omega_k+\delta)\frac{\Delta}{2}\right] e^{-j[(2\omega_k+\delta)\frac{\Delta}{2}]}} \quad (7)$$

$$T_{2_k}\left(\omega_k + \frac{\omega_s}{N}\right) = \frac{\sin\left[(2\omega_k+\delta)\frac{\Delta N}{2}\right] e^{(-j[(2\omega_k+\delta)\frac{\Delta N}{2}]}}{\sin\left[(2\omega_k+\delta)\frac{\Delta}{2} + \frac{\pi}{N}\right] e^{-j[(2\omega_k+\delta)\frac{\Delta}{2}+\frac{\pi}{N}]}} \quad (8)$$

The term δ represents the offset between the actual and estimated frequencies. On performing algebraic operations on Eqs. (3)–(8), the equations for the spectral peaks are obtained in terms of both $T_{1_k}(\omega)$ and $T_{2_k}(\omega)$:

$$\frac{2F_k(\omega_k - \omega_s/N)}{a_k e^{j\phi_c}} = T_{1_k}(\omega_k - \frac{\omega_s}{N}) + G^-(\delta) T_{2_k}(\omega_k) e^{-j2\phi_c} \quad (9)$$

$$\frac{2F_k(\omega_k)}{a_k e^{j\phi_c}} = T_{1_k}(\omega_k) + T_{2_k}(\omega_k) e^{-j2\phi_c} \quad (10)$$

$$\frac{2F_k(\omega_k + \omega_s/N)}{a_k e^{j\phi_c}} = T_{1_k}(\omega_k + \frac{\omega_s}{N}) + G^+(\delta) T_{2_k}(\omega_k) e^{-j2\phi_c} \quad (11)$$

The terms $G^-(\delta)$ and $G^+(\delta)$ are a shortened representation of the terms containing deterministic sinusoids. These terms are also a function of the offset δ and the signal length N.

$$G^-(\delta) = \left[\frac{C_2 \tan\frac{\delta\Delta}{2} + S_2}{C_1 \tan\frac{\delta\Delta}{2} + S_1}\right] e^{-j\frac{\pi}{N}}, \; G^+(\delta) = \left[\frac{C_2 \tan\frac{\delta\Delta}{2} + S_2}{C_3 \tan\frac{\delta\Delta}{2} + S_3}\right] e^{j\frac{\pi}{N}}$$

$$S_i = \sin\left[(2(k-1)+i)\frac{\pi}{N}\right], \quad C_i = \cos\left[(2(k-1)+i)\frac{\pi}{N}\right]$$

Further algebraic operations are carried out on Eqs. (9)–(11) to yield Eq. (12), resulting in the elimination of the negative frequency term for the given frequency that is to be estimated out of M frequencies:

$$\begin{aligned} &F_k^-[T_{1_k}^+(\delta) - G^+(\delta)] - F_k^+[T_{1_k}^-(\delta) - G^-(\delta)] + \\ &G^+(\delta)T_{1_k}^-(\delta) - G^-(\delta)T_{1_k}^+(\delta) = 0 \end{aligned} \quad (12)$$

where $F_k^{\pm}$ and $T_{1_k}^{\pm}$ are defined as:

$$F_k^{\pm} = \frac{F_k(\omega_k \pm \omega_s/N)}{F_k(\omega_k)} \quad (13)$$

$$T_{1_k}^{\pm}(\delta) = \frac{T_{1_k}(\omega_k \pm \omega_s/N)}{T_{1_k}(\omega_k)} \quad (14)$$

Equate the real part of Eq. (12) to zero to yield an expression which is a quartic (fourth degree) polynomial in the variable χ as shown:

$$\mathbb{P}_{\mathbb{k}}(\chi) = P_{0_k} + P_{1_k}\chi + P_{2_k}\chi^2 + P_{3_k}\chi^3 + P_{4_k}\chi^4 \quad (15)$$

where $\chi = \tan(\delta\Delta/2)$ represents the roots of the polynomial. The coefficients of the quartic polynomial corresponding to the frequency estimation of M sinusoids are real-valued and are derived as:

$$P_{0k} = S^2 S_2 \psi_{sk} \quad (16)$$

$$P_{1k} = S^2(S_2\psi_{ck} + C_2\psi_{sk}) + S(S_1 S_3 \gamma_k^+ - S_2\alpha^+) \quad (17)$$

$$\begin{aligned} P_2 &= S^2 C_2 \psi_{ck} - C^2 S_2 \psi_{sk} + S(C_3 S_1 + S_3 C_1)\gamma_k^+ + \\ &C(S_3 S_1 \gamma_k^- + S_2\alpha^-) - S(C_2\alpha^+ + S_2\beta^+) \end{aligned} \quad (18)$$

$$\begin{aligned} P_3 &= -C^2(S_2\psi_{ck} + C_2\psi_{sk}) + C(S_1 C_3 + C_1 S_3)\gamma_k^- + \\ &S(C_1 C_3 \gamma_k^+ - C_2\beta^+) + C(S_2\beta^- + C_2\alpha^-) \end{aligned} \quad (19)$$

$$P_4 = C(C_3 C_1 \gamma_k^- - CC_2 \psi_{ck} + C_2 \beta^-) \quad (20)$$

where the terms $\alpha^\pm$ and $\beta^\pm$ are deterministic constants. The terms $\psi_{ck}, \psi_{sk}, \gamma_k^\pm$ are random variables as they are a function of $R_k^\pm$ which comprise of terms containing the Fourier coefficients:

$$C = \cos(\pi/N),\ S = \sin(\pi/N)$$

$$\alpha^\pm = S_1 \pm S_3, \beta^\pm = C_1 \pm C_3$$

$$\psi_{ck} = C_1 R_k^- - C_3 R_k^+, \psi_{sk} = S_1 R_k^- - S_3 R_k^+$$

$$\gamma_k^\pm = R_k^- \pm R_k^+, R_k^\pm = Re[F_k^\pm e^{\mp j\frac{\pi}{N}}]$$

$$F_k^\pm = \frac{F_k(\omega_k \pm \omega_s/N)}{F_k(\omega_k)}$$

The unknown frequency is estimated from the solution of the quartic polynomial having real roots. Let $\chi = \chi_s$ be the solution of the quartic polynomial from which the unknown frequency is obtained as:

$$\omega_{ck} = \omega_{ik} + \delta \quad (21)$$

where

$$\delta = \frac{2}{\Delta}\tan^{-1}(\chi_s) \quad (22)$$

χ_s is given as $\chi_s \in [-\tan(\pi/N), \tan(\pi/N)]$.

The solution of the polynomial that lies in the bin corresponding to the peak value is used to compute the frequency $\chi = \tan(\delta\Delta/2)$ estimate and all other roots are neglected. The analytical method available for finding the solution of the polynomial is available in [24]. Since the exact analytical solutions of polynomials are known up to fourth degree, the frequency estimation problem can be solved efficiently with the help of a computing platform devoid of any iterative or computationally intensive operations.

B. Peak Detection in Multi-Frequency Estimation

The spectrum of a noiseless multi-frequency sinusoidal signal consists of as many dominant peaks as the number of frequencies that constitute the signal. We adopt a simple technique of detecting the peaks by using the gradient based approach. For the peak detection problem in multi-frequency estimation, we assume that the model order M corresponding to M frequencies in the multi-sinusoidal signal is known beforehand. The spectrum of the composite signal is obtained by computing the FFT of the signal.

From the spectral information, we identify the largest peak from the existing peaks based on the gradient computation of the slopes. Determine the two adjacent frequencies corresponding to the peak frequency ω_{ik} as the centre, to obtain ω_{ik}, $\omega_{ik} - \omega_s/N$ and $\omega_{ik} + \omega_s/N$ and proceed to perform the frequency estimation corresponding to the maximum peak using the quartic polynomial approach explained in the earlier subsection. Intuitively, based on the choice of the sampling interval, the frequency term ω_{ik} falls in the bin of interest which also corresponds to the bin in which the root of the quartic polynomial is present. For the estimation of frequencies, corresponding to the subsequent peaks, the maximum peak is discarded from the spectrum to negate the effect of the estimated peak. From determining the adjacent frequencies, the same processes are repeated on the subsequent peaks till all the M peaks are obtained. This gradient based approach to detection of peaks in the spectrum is possible because of the spectral resolution of DFT ensuring that the sinusoids are separated by at least one DFT bin. As each frequency is estimated, the root mean square error (RMSE) is calculated to ascertain the efficacy of the estimator.

C. Complexity Analysis of Multi-Frequency Estimation

The complexity of the proposed approach is small since the frequency estimates are obtained from the roots of the fourth-order polynomial. By virtue of the problem statement and the formulation of the estimation technique, the method turns out to be non-iterative compared to several existing formulations which are mostly iterative. The notion of efficiency of the proposed method is stated in terms of the computational complexity involved.

To the best of author's knowledge, the number of arithmetic operations required in all the previously reported works is a function of the signal length N. The complexity is calculated considering the operations required to perform FFT, complex and real valued multiplication, addition, and other associated mathematical operations. An exhaustive comparison of various methods as well as the computational efficiency was discussed in [25]. A direct extension of the method to multifrequency estimation will require greater than $16MN\log_2(N)+50MN$ arithmetic operations at the very least, where M denotes the model order or the number of sinusoids. Similarly, several other methods as can be seen in [25] have a strong dependency on N. This implies that as the signal length increases, the computational complexity will also increase proportionally.

Since the proposed method relies only on the knowledge of the analytical solution of the quartic polynomial, the algorithm does not directly exhibit a dependency on the signal length. Considering the operations involving the computation of the root falling in the desired DFT bin and the simple arithmetic operations involved in computing the desired root, the number of arithmetic operations involved is $\approx 70M$. Hence, the proposed algorithm offers much lesser computational complexity and is desirable for implementation in real-time contexts.

IV. Results and Discussion

This section discusses the results of multi-frequency estimation of real sinusoids using the quartic polynomial approach. The sampled signal consists of a combination of two or more frequencies which are subject to estimation by the proposed algorithm. We study the accuracy of estimation of our method by computing the root mean square error (RMSE) for varying SNR values. In all simulations, samples of real sinusoid with multiple frequencies embedded in noise are considered. The phase of the sinusoid is uniformly distributed within the interval [0, 2π]. The obtained plots are compared with the Cramer Rao Lower Bound (CRLB), which provides a theoretical lower bound on the achievable estimation error for the frequencies. For the case of real sinusoids, where the SNR of the kth real sinusoid is given as $\text{SNR}_k = A_k^2/2\sigma^2$, the expression for the CRLB is given as,

$$\text{var}(\omega_k) \geq \frac{12}{\text{SNR}_k N(N^2-1)} \tag{23}$$

Fig. 2 shows the RMSE performance of the quartic polynomial based algorithm for the estimation of f_1. The results are compared with those obtained in [7], by adopting the same simulation settings. We perform the simulation for M = 2 and the frequency values are chosen as f_1 = 0.15 and f_2 = 0.35. In accordance with the parameters in [7], the amplitude values are chosen as $A_1 = \sqrt{2}$ and $A_2 = \sqrt{2}/2$. The number of samples corresponding to (1) is taken as N = 200. It is observed that, compared to the Linear Prediction approach of [7] with the same simulation parameters, the quartic polynomial based approach converges to CRLB faster.

Extending the simulation experiments based on the same parameter settings as in [7], Fig. 3 and Fig. 4 depict the RMSE performance of the proposed algorithm for varying signal lengths ranging from N = 32 to N = 1024. The simulations are carried out for low and moderate SNR values ranging between -3 dB to 6 dB. Fig. 3 shows the RMSE performance for the estimation of f_1 and Fig. 4 presents the performance of the algorithm for the estimation of f_2.

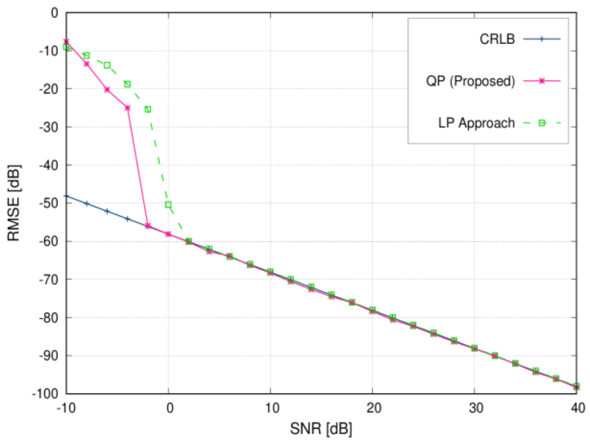

Fig. 2. Comparison of RMSE performance of the proposed approach with LP method for the estimation of f_1.

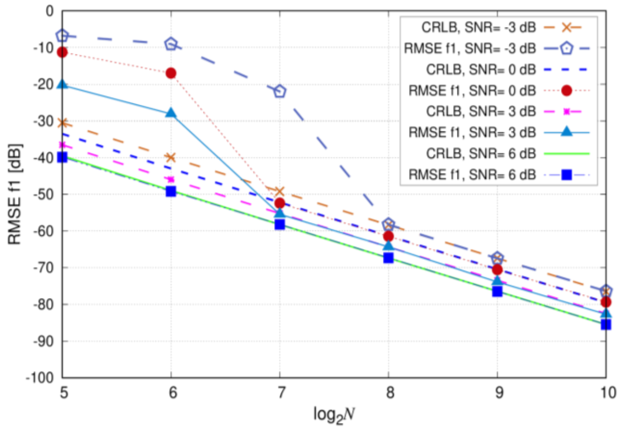

Fig. 3. RMSE performance of quartic polynomial based multi-sinusoidal frequency estimation for varying signal length for the estimation of f_1.

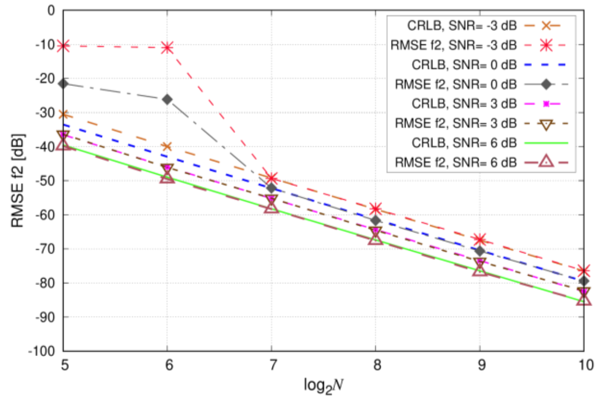

Fig. 4. RMSE performance of quartic polynomial based multi-sinusoidal frequency estimation for varying signal length for the estimation of f_2.

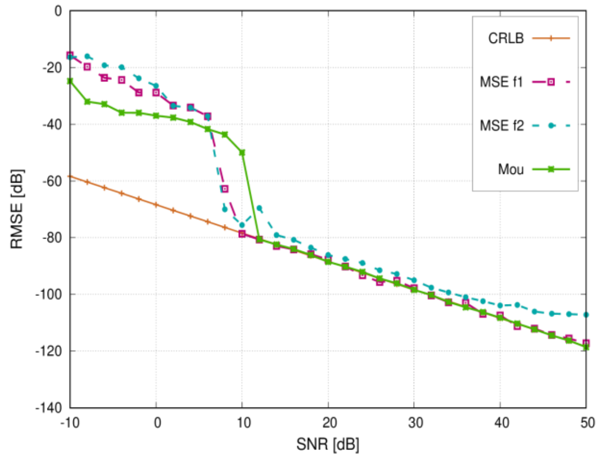

Fig. 5. RMSE performance of quartic polynomial based multi-sinusoidal frequency estimation considering five frequencies for varying signal length for the estimation of f_1.

We also evaluate the performance of the algorithm for model order M=5. The RMSE performance for the estimation of f_1 is shown in Fig. 5. The efficacy of the algorithm is compared with [11]. In accordance with the sinusoidal parameters in [11], the frequency values are chosen as f_1 =0.05, f_2 =0.13, f_3 =0.26, f_4 = 0.34 and f_5 = 0.44. It is evident from Fig. 5 that even for sufficiently high model order, that is, in the presence of 5 frequencies, the estimation of the frequency corresponding to the dominant peak exhibit good performance, conforming to the CRLB.

We have also presented the performance of estimation of f_2 along with f_1. For the estimation of f_1, the performance of the algorithm meets the CRLB from SNR=10 dB onwards, establishing the efficacy of the algorithm for moderate and high SNR values. In comparison with [11], although the quartic polynomial algorithm is slightly sub-optimal at low SNR values, it is observed that the algorithm converges earlier.

In the context of multi-sinusoidal frequency estimation, it is significant to evaluate the performance of the algorithm in terms of the separation between the frequency components. It is a measure of how accurately frequency estimation can be performed when the subsequent frequencies are spaced close together.

In Fig. 6, we consider a sinusoid consisting of two components with frequencies f_1 and $f_2 = f_1 + \Delta f$. In this simulation environment, we consider M = 2, i.e, two sinusoids. The parameters chosen for simulation are a_1 = 1, a_2 = 0.8, N = 256 and SNR (dB) = 10 dB. The term Δf represents the frequency displacement. Fig. 6 presents the RMSE performance of the estimation of f_1 and f_2 evaluated for varying values of normalized frequency displacement ΔfN. The simulation is performed for the interval $[1/N : 0.25]$, considering increments in the step of $1/N$. In each iteration, f_1 is chosen randomly from the interval (–0.5, 0.5). It is observed that in the estimation of f_1, the estimation error is lower than the errors in estimation of f_2.

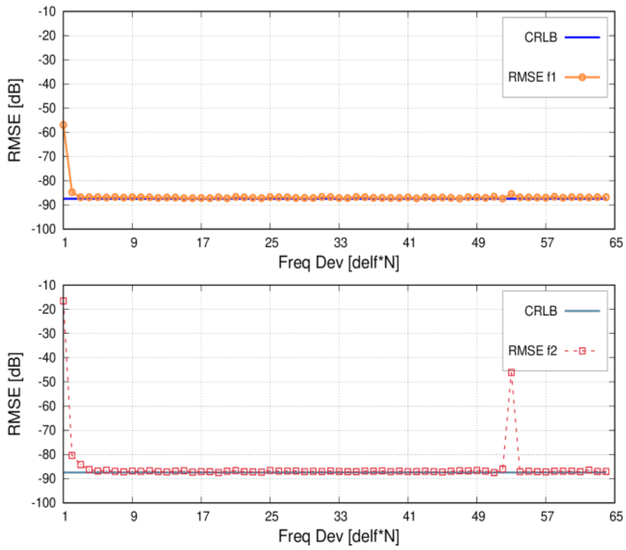

Fig. 6. Performance of the estimation of frequencies in a multiple sinusoid with M = 2 versus the frequency displacement.

At very low values of frequency deviation, the resolution of the frequencies is less than one DFT bin, which does not align with the assumption considered in the simulation. In general, at very small values of frequency displacements, the performance of algorithm deviates from the CRLB. For all other values of frequency deviation, the performance of the estimator closely follows the CRLB. This is indicative of the capability of the algorithm to be able to resolve the peaks in the spectrum.

V. Conclusion

In this work, we present the quartic polynomial approach for the frequency estimation of multiple real sinusoids in additive white Gaussian noise. A simple gradient-based approach is adopted to identify the peaks and subsequently perform the frequency estimation from the spectral information. Firstly, the spectrum of the composite signal is evaluated, and all the peaks are identified. Starting from the dominant peak, the peaks are estimated one by one. Each estimated peak is removed before the next dominant peak is identified. This is continued till all the peaks are obtained and then the quartic polynomial based frequency estimation for multiple sinusoids is implemented for the estimation of each of the frequencies by presenting the derivations as applicable for the multi-sinusoidal estimation.

The performance of the algorithm is studied by computing the estimation error for varying signal-to-noise ratio, varying values of signal length and frequency deviation. Simulation studies are presented to establish the validity of the estimation algorithm for multiple sinusoids, by comparing the obtained results against existing works. The proposed approach is fast and efficient, rendering it suitable for application to real-time implementations.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

Conceptualization of the work is to the credit of Dr. Dhanesh G. Kurup. Gayathri carried out an extensive literature survey in the field of multi-sinusoidal frequency estimation for real sinusoids. Dr. Kurup and Gayathri derived the equations for polynomial based frequency estimation. The simulation results were generated by Gayathri and verified by Dr. Kurup. The paper was written by Gayathri and proof-reading was done by Dr. Kurup. All authors approved of the final version.

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