Re-Evaluating the SPRT Chart with Estimated Process Parameters When the Underlying Distributions Are Gamma, Lognormal, and Weibull

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***Abstract*—To reduce the impact of Phase-I parameter estimation on the performances of Phase-II control charts, researchers have incorporated the ideology of Guaranteed In-control Performance (GICP) in their statistical designs to limit the risk of excessive false alarms. At present, most research works have primarily focused on normally distributed data. However, the assumption of normality is often violated in manufacturing environments, and certain data may exhibit positively skewed distributions. In this paper, we investigate the performance of the SPRT control chart with estimated process parameters designed using the GICP method under three different skewed distributions, i.e., the Gamma, Lognormal, and Weibull distributions. The study is conducted by varying the Phase-I sample size and the degree of skewness in order to reveal their impacts upon the in-control and out-of-control performances of the SPRT chart with estimated process parameters. Results show that an increase in the skewness level leads to rapid deterioration in both the in-control and out-of-control expected values of the average time to signal (AATS) and the average standard deviation of the time to signal (ASDTS). Interestingly, we have found that increasing the Phase-I sample size leads to deterioration in the conditional in-control performance, but an improvement in the out-of-control AATS and ASDTS values. Furthermore, it is found that, among the three distributions, the Lognormal distribution produces the least stable performance when skewness is large and the Phase-I sample size is small.**

***Index Terms*—Average time to signal, guaranteed in-control performance, parameter estimation, sequential probability ratio test, skewed distribution, statistical process monitoring**

I. Introduction

Statistical Process Monitoring (SPM) is widely practiced in various industries to control the stability of industrial processes as well as to maintain the quality of production outputs. The control chart, which is one of the most common tools in SPM, has garnered significant attention due to its operational simplicity and user-friendly interface. Generally, time-weighted control schemes are favored over traditional Shewhart schemes as they respond quicker to small and moderate deviations in the process quality characteristic. The sequential probability ratio test (SPRT) control chart, in particular, stands out as a popular example of time-weighted control schemes.

Stoumbos & Reynolds [1] constructed the SPRT chart by applying independent sequential tests at fixed time intervals over the course of process monitoring. Each test samples a random number of observations and is expected to conclude in negligible time compared to the time between successive tests. Many research findings have demonstrated that the SPRT chart is more effective than the Shewhart and cumulative sum (CUSUM) charts in detecting various magnitudes of process shifts [2–4]. Ou *et al.* [5] proposed an optimization design for the SPRT chart based on the average extra quadratic loss (AEQL) criterion. The AEQL is a weighted measure of the overall performance of a control chart over a specified range of shifts. They showed that, by minimizing the AEQL over a range of shift sizes, the SPRT chart enjoys a generally short average time to signal (ATS) at each shift point within the range. Haridy *et al.* [6] suggested to optimize the in-control average sample number (ASN) and the reference parameter of the SPRT chart to further boost its average detection speed. They showed that the proposed optimal design effectively reduces the detection time of the SPRT chart by almost twice. Following these successful attempts, Mahadik & Godase [4] and Godase & Mahadik [7] developed the SPRT sign charts for monitoring process mean and process variance, respectively, by applying the sequential sign test. Findings have shown that the SPRT sign chart has a superior performance over other competing charts, such as the synthetic Shewhart sign, CUSUM sign, and EWMA sign charts.

Manuscript received August 1, 2023; revised September 3, 2023; accepted September 13, 2023; published February 2, 2024.
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Industrial process monitoring can be divided into two main phases, i.e., Phase-I and Phase-II. In Phase-I, a moderate set of samples is collected to verify the stability of the manufacturing process. These Phase-I data are then used to estimate the process parameters, i.e., mean and standard deviation. In Phase-II, practitioners usually set these parameter estimates as the target process parameters, which are then used to evaluate the status of the ongoing process until an out-of-control event emerges. It is worth noting that, although the samples come from the same in-control source, the parameter estimates can vary by a considerable degree. This phenomenon, known as practitioner-to-practitioner variability, is usually much more prominent when the number of Phase-I samples is small. In standard designs of the control charts, researchers often neglect the reality that the process parameters are estimated. This results in substantial variability in the chart's performances, as well as unacceptably high false alarm levels. Teoh *et al.* [8] found out that there is a near 50% chance that a practitioner will obtain a false alarm rate higher than the recommended rate, when the control charts with estimated process parameters are designed via the traditional approach. Similar findings have been obtained by [9–11]. To rectify this issue, the Guaranteed In-control Performance (GICP) framework has been introduced. In this approach, the charting limits are adjusted to ensure that the conditional in-control performance exceeds a specific threshold with a very high probability (e.g., 90% or 95%). Teoh *et al.* [8] and Diko *et al.* [9] have reported that the GICP approach effectively reduces the risk of excessive false alarms associated with Phase-I parameter estimation, while retaining a satisfactory level of out-of-control performance. This approach is extensively used in the statistical designs of a myriad of control charts, see for example, [12, 13]. For a comprehensive review of mainstream control charts with estimated process parameters, refer to [14].

The majority of control charts in the literature are built upon the assumption that process data come from the Normal distribution. However, in certain industries, the process quality characteristics can exhibit highly skewed distributions. Some examples include tensile strength measurements of glass fibers [15], lifetimes in accelerated life test samples [16], and durations to the detection of urinary tract infection [17]. Several researchers, such as [18–20], have pointed out that positively skewed data can increase the likelihood of false alarms in Phase-II chart applications. This issue is exacerbated when the control chart is operating with estimated process parameters under the traditional design. While the GICP framework has proven ability in controlling the risk of false alarms, there are some concerns regarding its effectiveness when dealing with skewed process data. To the best of our knowledge, almost none of the existing works have investigated the impact of skewness on the performance of GICP-adjusted control charts. Therefore, in this paper, we examine the performance of the GICP-adjusted optimal SPRT chart developed by Teoh *et al.* [8] under various skewness levels. We consider three commonly used skewed distributions, i.e., the Gamma, Lognormal, and Weibull distributions.

The organization of this paper is as follows. In the methodology section, we first describe the charting structure and procedure of the SPRT chart. It is then followed by a brief review of the run-length properties of the SPRT chart under the Normal distribution, both with known and estimated process parameters. In the case of known parameters, formulae for the ATS, standard deviation of the time to signal (SDTS), and ASN are provided; whereas in the case of estimated parameters, formulae for the average of the ATS (AATS), average of the SDTS (ASDTS), and the average of the ASN (AASN) are provided. We also detail the statistical properties of the Gamma, Lognormal, and Weibull distributions. In the methodology section, we describe the design of our experiment and outline the measures taken to ensure fairness throughout our comparative study. In the results section, we present tables showing the AATS and ASDTS values of the optimal SPRT chart with estimated process parameters under the three skewed distributions. Finally, we summarize our findings and provide some concluding remarks.

II. Background

A. The SPRT Chart under the Normal Distribution

Let Y be the quality characteristic of a normally distributed process such that its in-control mean and variance are equal to μ_0 and σ_0^2, respectively. Suppose that we are interested in detecting a change in the process mean, i.e., from $\mu = \mu_0$ to $\mu = \mu_0 + \delta\sigma_0$, where μ is the mean of the ongoing process and δ is the standard mean shift size. Suppose further that we are only interested in detecting an upper-sided mean shift, i.e., δ>0. The charting statistic ($U_{i,j}$) of the upper-sided SPRT chart is

$$U_{i,j} = \sum_{\theta=1}^{j}\left(\frac{Y_{i,\theta} - \mu_0}{\sigma_0} - \gamma\right), \qquad (1)$$

for i=1, 2, ···, and j=1, 2, ···, N_i, where N_i is the total number of measurements until the ith SPRT is terminated. In Eq. (1), $Y_{i,\theta}$ is the θth measurement of the ith SPRT and $\gamma > 0$ is a reference parameter. It is assumed that the observations $Y_{i,\theta}$ are sampled independently in an insignificant amount of time for each SPRT.

The SPRT chart has two control limits, i.e., a lower control limit g and an upper control limit h. During the sampling process,

- if $U_{i,j} < g$, the process is indicated as in-control, and sampling is terminated,
- if $U_{i,j} > h$, the process is indicated as out-of-control, and sampling is terminated,
- if $g \le U_{i,j} \le h$, the status of the process cannot be determined, and sampling resumes.

It should be noted that, when the process is signaled as out-of-control, a designated out-of-control action plan is executed to identify the root causes of the process shift.

B. The Run-length Properties of the SPRT Chart with Known and Estimated Process Parameters

When process parameters are known, we can derive the formulae for the ASN, ATS, and SDTS of the SPRT chart by means of the Markov chain approach [5]. The proof begins by partitioning the region $[g, h]$ into a large number of subintervals, say ξ subintervals. Each subinterval is regarded as a transient state of the Markov chain. The ASN of the SPRT chart is computed as

$$\text{ASN} = 1 + \mathbf{C}^{\text{T}}(\mathbf{I} - \mathbf{R})^{-1}\mathbf{1}, \quad (2)$$

where $\mathbf{R}$ is a $\xi\times\xi$ transition probability matrix, with entries denoted as $r_{s,t} = \Phi[\Delta(t-s+0.5)+\gamma-\delta] - \Phi[\Delta(t-s-0.5)+\gamma-\delta]$, $\mathbf{C}$ is a $\xi\times1$ vector, with entries denoted as $c_s = \Phi[\Delta s+g+\gamma-\delta] - \Phi[\Delta(s-1)+g+\gamma-\delta]$, $\mathbf{I}$ is the $\xi\times\xi$ unit matrix, and $\mathbf{1}$ is a $\xi\times1$ vector filled with ones. Here, $\Delta=(h-g)/\xi$ is the width of each subinterval, and $\Phi(\cdot)$ is the cumulative distribution function (cdf) of the standard normal distribution $N(0, 1)$.

The ATS and SDTS of the SPRT chart are evaluated as

$$\text{ATS} = d\left(\frac{1}{1-\text{OC}(\delta)} - \frac{1}{2}\mathbb{F}\{\delta \neq 0\}\right) \quad (3)$$

and

$$\text{SDTS} = d\sqrt{\frac{\text{OC}(\delta)}{[1-\text{OC}(\delta)]^2} + \frac{1}{12}\mathbb{F}\{\delta \neq 0\}}, \quad (4)$$

respectively, where $d>0$ is the sampling interval, $\mathbb{F}(\cdot)$ is the indicator variable, and $\text{OC}(\delta)=P+\mathbf{C}^{\text{T}}(\mathbf{I}-\mathbf{R})^{-1}\mathbf{Q}$ is the operating characteristic function of a single SPRT. Here, $P=\Phi(g+\gamma-\delta)$ is the probability that the process is accepted as in-control after the first observation, and $\mathbf{Q}$ is a $\xi\times1$ vector, with entries denoted as $q_s=\Phi[\Delta(0.5-s)+\gamma-\delta]$. Note that the operating characteristic function represents the probability that the process is declared as in-control, given that the true mean shift is δ.

Teoh *et al.* [8] developed the full theoretical framework for the SPRT chart with estimated process parameters. Assuming that the process parameters are unknown, we estimate μ_0 and σ_0 using a set of in-control Phase-I samples of size m, i.e., $X_1, X_2, \cdots, X_m$, as

$$\hat{\mu}_0 = \frac{1}{m}\sum_{\theta=1}^{m} X_\theta \quad (5)$$

and

$$\hat{\sigma}_0 = \sqrt{\frac{1}{m-1}\sum_{\theta=1}^{m}(X_\theta - \hat{\mu}_0)^2}, \quad (6)$$

respectively.

Teoh *et al.* [8] showed that the ASN, ATS, and SDTS of the SPRT chart with estimated process parameters are now conditional functions of the random variables $\hat{\mu}_0$ and $\hat{\sigma}_0$. By constructing the pivotal quantities $W=(\hat{\mu}_0 - \mu_0)/(\sigma_0/\sqrt{m})$ and $V=\hat{\sigma}_0/\sigma_0$, they derived the conditional ASN (CASN), conditional ATS (CATS), and conditional SDTS (CSDTS) as

$$\text{CASN} = 1 + \hat{\mathbf{C}}^{\text{T}}(\mathbf{I} - \hat{\mathbf{R}})^{-1}\mathbf{1}, \quad (7)$$

$$\text{CATS} = d \times \left(\frac{1}{1-\text{OC}(\delta)} - \frac{1}{2}\mathbb{F}\{\delta \neq 0\}\right), \quad (8)$$

and

$$\text{CSDTS} = d \times \left\{\frac{1+\text{OC}(\delta) - [\text{OC}(\delta) - \text{OC}^2(\delta)]\mathbb{F}\{\delta \neq 0\}}{[1-\text{OC}(\delta)]^2} - \frac{2}{3}\mathbb{F}\{\delta \neq 0\} - \left(\frac{\text{CATS}}{d}\right)^2\right\}^{0.5}, \quad (9)$$

respectively, where $\text{OC}(\delta) = \hat{P} + \hat{\mathbf{C}}^{\text{T}}(\mathbf{I} - \hat{\mathbf{R}})^{-1}\hat{\mathbf{Q}}$ is the updated operating characteristic function. Here, the entries of $\hat{\mathbf{R}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{Q}}$ are updated as $\hat{r}_{s,t} = \Phi\{V[\Delta(t-s+0.5)+\gamma]-\delta+W/\sqrt{m}\} - \Phi\{V[\Delta(t-s-0.5)+\gamma]-\delta+W/\sqrt{m}\}$, $\hat{c}_s = \Phi[V(\Delta s+g+\gamma)-\delta+W/\sqrt{m}] - \Phi\{V[\Delta(s-1)+g+\gamma]-\delta+W/\sqrt{m}\}$, and $\hat{q}_s = \Phi\{V[\Delta(0.5-s)+\gamma]-\delta+W/\sqrt{m}]$, respectively; whereas $\hat{P}$ is updated as $\Phi[V(g+\gamma)-\delta+W/\sqrt{m}]$.

To evaluate the unconditional performances of the SPRT chart with estimated process parameters, Teoh *et al.* [8] derived expressions for the AASN, AATS, and ASDTS as

$$\text{AASN} = \int_0^\infty \int_{-\infty}^\infty \text{CASN}\, f_W(w) f_V(v) dw dv, \quad (10)$$

$$\text{AATS} = \int_0^\infty \int_{-\infty}^\infty \text{CATS}\, f_W(w) f_V(v) dw dv, \quad (11)$$

and

$$\text{ASDTS} = d \times \left\{\int_0^\infty \int_{-\infty}^\infty \frac{1+\text{OC}(\delta) - [\text{OC}(\delta) - \text{OC}^2(\delta)]\mathbb{F}\{\delta \neq 0\}}{[1-\text{OC}(\delta)]^2} f_W(w) f_V(v) dw dv - \frac{2}{3}\mathbb{F}\{\delta \neq 0\} - \left(\frac{\text{AATS}}{d}\right)^2\right\}^{0.5}, \quad (12)$$

respectively, where $f_W(w)$ and $f_V(v)$ are the probability density functions (pdf) of the random variables W and V, respectively. When the process follows the Normal distribution, W follows $N(0, 1)$; whereas V follows a distribution whose pdf is $f_V(v)=2(m-1)v\, f_{\chi^2,m-1}[(m-1)v^2]$. Here, $f_{\chi^2,m-1}(\cdot)$ is the pdf of the Chi-squared distribution with $m-1$ degrees of freedom. Note that (10) to (12) can be thought of as the "average" of (7) to (9) over all parameter estimates.

C. Statistical Properties of the Gamma, Lognormal, and Weibull Distributions

In this section, we present the statistical properties, i.e., the skewness (k), the in-control mean ($\mu_{Z,0}$) and the in-control standard deviation ($\sigma_{Z,0}$), of the Gamma, Lognormal, and Weibull distributions. These distributions are chosen because their parameters can be tuned to achieve a multitude of shapes and skewness [16].

The cdf of the single-parameter Gamma distribution is given as $F_Z(z) = \int_0^z t^{\alpha-1}\exp(-t)dt/\Gamma(\alpha)$ for $z\geq 0$, where $\Gamma(\cdot)$ is the gamma function and $\alpha>0$ is the shape parameter. The skewness, in-control mean, and standard deviation can be computed as [21]

$$k = \frac{2}{\sqrt{\alpha}}, \tag{13}$$

$$\mu_{Z,0} = \alpha , \tag{14}$$

$$\sigma_{Z,0} = \sqrt{\alpha} , \tag{15}$$

respectively.

The cdf of the two-parameter Lognormal distribution is given as $F_Z(z)=\Phi[(\ln z-\mu_{LN})/\sigma_{LN}]$ for $z>0$, where μ_{LN} and $\sigma_{LN}>0$ are the location and scale parameters, respectively. For the ease of computation, we set $\mu_{LN}=0$, since the value of μ_{LN} does not affect the skewness of the Lognormal distribution. The skewness, in-control mean, and standard deviation are evaluated as [22]

$$k = [2+\exp(\sigma_{LN}^2)]\sqrt{\exp(\sigma_{LN}^2)-1} , \tag{16}$$

$$\mu_{Z,0} = \exp\left(\frac{1}{2}\sigma_{LN}^2\right), \tag{17}$$

$$\sigma_{Z,0} = \sqrt{\exp(\sigma_{LN}^2)[\exp(\sigma_{LN}^2)-1]} , \tag{18}$$

respectively.

The cdf of the two-parameter Weibull distribution is given as $F_Z(z)=1-\exp[-(\lambda z)^\alpha]$ for $z\geq 0$, where $\beta>0$ and $\lambda>0$ are the shape and scale parameters, respectively. For convenience, we set $\lambda=1$ throughout this paper. The skewness, in-control mean, and standard deviation have the following formulae [23]

$$k = \frac{\Gamma\left(1+\frac{3}{\beta}\right)+2\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^3-3\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1+\frac{2}{\beta}\right)}{\left\{\Gamma\left(1+\frac{2}{\beta}\right)-\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\}^{1.5}}, \tag{19}$$

$$\mu_{Z,0} = \Gamma\left(1+\frac{1}{\beta}\right), \tag{20}$$

$$\sigma_{Z,0} = \sqrt{\Gamma\left(1+\frac{2}{\beta}\right)-\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2} , \tag{21}$$

respectively.

III. Methodology

In our experiment, we design the SPRT chart with estimated process parameters using the GICP method. The formulation of the GICP is as follows:

$$\Pr[\text{CATS}_0 \geq \tau(1-\varepsilon)] = 1-p , \tag{22}$$

where CATS_0 is the in-control CATS (i.e., obtained by substituting $\delta=0$ in (8)), τ is the minimum acceptable in-control ATS (ATS_0), ε is a small tolerance term, and p is a user-specified error probability. To limit the risk of false alarms, p is usually specified as a small percentage, e.g., 90% or 95%. It is important to note that increasing the value of ε and/or p in (22) can lead to a higher risk of obtaining undesirable levels of false alarms, but at the same time, it improves the out-of-control performance of the control chart towards small process shifts [9]. Note that the design is implemented under the assumption that the data follow the Normal distribution.

To achieve the best overall performance, as well as to minimize the trade-off between the in-control and out-of-control performances, Teoh *et al.* [8] introduced an integrated GICP-optimal design for the SPRT chart with estimated process parameters. They proposed minimizing the average of the AEQL (AAEQL) over a range of mean shifts [$\delta_{\min}$, $\delta_{\max}$], subject to the GICP constraint stated in (22). The formula for the AAEQL is

$$\text{AAEQL} = \frac{1}{\delta_{\max}-\delta_{\min}} \times \int_0^{\infty}\int_{-\infty}^{\infty}\int_{\delta_{\min}}^{\delta_{\max}} \delta^2 \text{CATS}\, f_W(w) f_V(v) d\delta dw dv , \tag{23}$$

where CATS can be quoted directly from (8).

To ensure fairness in our comparative study, we impose the following five specifications for all the SPRT charts designed in this paper: the minimum mean shift size $\delta_{\min}$, maximum mean shift size $\delta_{\max}$, minimum sampling interval allowed $d_{\min}$, the inspection rate R, and τ. The values of $\delta_{\min}$ and $\delta_{\max}$ are set according to the practitioner's knowledge about the degree of departure of the process from its usual level. $d_{\min}$ should be set as a suitable value according to the factory's inspection policy. The inspection rate R, which is defined as the ratio of the in-control AASN (AASN_0) to the sampling interval d, should be selected based on practical considerations such as the availability of manpower. The value of τ is set to give a reasonably small false alarm probability. In this paper, we set the design specifications as ($\delta_{\min}$, $\delta_{\max}$, $d_{\min}$, R, τ)=(0.1, 3.0, 0.25, 3, 370.40). Note that τ=370.40 is set to give an average false alarm probability (=0.27%) equivalent to that of the Shewhart $\bar{X}$ chart with six-sigma limits under the Normal distribution. The optimal SPRT chart with estimated process parameters is designed by minimizing the AAEQL over the shift range [$\delta_{\min}$, $\delta_{\max}$], subject to four constraints, i.e., $\Pr[\text{CATS}_0 \geq \tau(1-\varepsilon)]=1-p$, $R=\text{AASN}_0/d$, $\text{AASN}_0>1$, and $d>d_{\min}$. This involves searching the charting parameters (AASN_0, γ, d, g, h) in their feasible ranges in order to minimize the objective function, while keeping the constraints satisfied. The full optimization algorithm can be found in Teoh *et al.* [8], and is omitted in this paper for the purpose of brevity.

IV. Results and Discussions

Table I shows the optimal charting parameters ($AASN_0$, γ, d, g, h) of the SPRT chart with estimated process parameters designed for Phase-I sample sizes $m \in \{200, 400, 600, 1000, +\infty\}$, along with the (AATS, ASDTS) values for $\delta \in \{0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}$. The specifications for the GICP method are chosen as p=0.05 and ε=0.2. Note that m=+∞ corresponds to the case where the process parameters are known, hence no parameter estimation is required. Therefore, Eqs. (3) and (4) are used to compute the ATS and SDTS values of the optimal SPRT chart when m=+∞. All the other results are computed using (11) and (12), and they have been verified using Monte Carlo simulation with 100,000 simulation runs. As a numeric example, when m=400 Phase-I samples are used, the charting parameters of the SPRT chart with the best AAEQL performance are ($AASN_0$, γ, d, g, h)=(1.597, 0.332, 0.532, 0.686, 8.634). The (AATS, ASDTS) values at δ=1.0 are equal to (1.06, 1.05), respectively.

Table II display the in-control (AATS, ASDTS) values (i.e., ($AATS_0$, $ASDTS_0$)) of the optimal SPRT chart with GICP-adjusted limits; whereas Table III, Table IV, and Table V display the out-of-control (AATS, ASDTS) values (i.e., ($AATS_1$, $ASDTS_1$)) under the Gamma, Lognormal, and Weibull distributions, respectively. Due to the complexity of the sampling distributions of the Gamma, Lognormal, and Weibull distributions (i.e., the joint density of W and V), it is not feasible to compute the AATS and ASDTS values using (11) and (12). Hence, we resort to Monte Carlo simulation with 100,000 runs to approximate the performances of the SPRT chart under these skewed distributions.

Table II tabulates three in-control performance indicators, i.e., the in-control exceedance probability $\Pr(CATS_0 \geq \tau)$, $AATS_0$, and $ASDTS_0$ values, of the optimal SPRT chart with GICP-adjusted limits for skewness $k \in \{0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}$. The table is separated into four layers: the first layer shows the baseline performance under the Normal distribution, whereas the second, third, and fourth layers show the performances under the Gamma, Lognormal, and Weibull distributions, respectively. The values of the parameters (i.e., α, σ_{LN}, and β) affecting the skewness of each distribution can be computed using standard root-finding methods. For example, to achieve a skewness of 3.0 for the Lognormal distribution, we can solve (16) by setting k=3.0, which results in σ_{LN}=0.71557. The charting parameters used to compute the results for each $m \in \{200, 400, 600, 1000, +\infty\}$ can be quoted directly from Table I. As a numeric example, when m=200, k=0.5, and the underlying distribution is Weibull, the SPRT chart with charting parameters ($AASN_0$, γ, d, g, h)=(1.630, 0.386, 0.543, 0.540, 8.559) yields $\Pr(CATS_0 \geq \tau)$=83.30% and ($AATS_0$, $ASDTS_0$)=(2695.90, 9535.27).

From Table II, it is observed that when k=0.0, the exceedance probabilities $\Pr(CATS_0 \geq \tau)$ for the Gamma, Lognormal, and Weibull distributions are fairly close to that of the Normal distribution for all values of m, when the process parameters are estimated. We notice that $\Pr(CATS_0 \geq \tau)$ for the Normal distribution is slightly lower than 95%. This is because a tolerance term of ε=0.2 has been introduced in the design of the SPRT chart, hence lowering the threshold for the GICP constraint (refer to (22)). Referring to the unconditional performances of the three distributions under k=0.0, it is noticed that the Weibull distribution produces slightly larger ($AATS_0$, $ASDTS_0$) values compared to the Gamma, Lognormal, and Normal distributions. For example, when m=600 and k=0.0, the ($AATS_0$, $ASDTS_0$) values of the Gamma (=(1084.11, 1639.53)) and Lognormal (=(1084.30, 1635.58)) distributions are rather close to those of the Normal distribution (=(1091.10, 1638.63)); whereas the ($AATS_0$, $ASDTS_0$) values of the Weibull distribution (=(1131.91, 1728.81)) are arguably higher than the rest.

TABLE I: OPTIMAL CHARTING PARAMETERS ($AASN_0$, γ, d, g, h), AATS, AND ASDTS VALUES OF THE SPRT CHART WITH ESTIMATED PROCESS PARAMETERS DESIGNED UNDER THE GICP FRAMEWORK, FOR $m \in \{200, 400, 600, 1000, +\infty\}$, τ = 370.4, R = 3, d_{min} = 0.25, δ_{min} = 0.1, δ_{max} = 3, p = 0.05, AND ε = 0.2

m	200	400	600	1000	+∞
	($AASN_0$, γ)	($AASN_0$, γ)	($AASN_0$, γ)	($AASN_0$, γ)	(ASN_0, γ)
	(d, g, h)	(d, g, h)	(d, g, h)	(d, g, h)	(d, g, h)
δ	(AATS, ASDTS)	(AATS, ASDTS)	(AATS, ASDTS)	(AATS, ASDTS)	(ATS, SDTS)
	(1.630, 0.386)	(1.597, 0.332)	(1.624, 0.344)	(1.632, 0.363)	(1.587, 0.380)
	(0.543, 0.540, 8.559)	(0.532, 0.686, 8.634)	(0.541, 0.618, 7.991)	(0.544, 0.553, 7.297)	(0.529, 0.541, 6.327)
0.0	(5856.56, > 20000)	(1784.91, 3501.03)	(1091.10, 1638.63)	(724.57, 901.03)	(370.40, 370.13)
0.5	(6.66, 9.59)	(5.02, 5.54)	(4.79, 5.09)	(4.72, 4.89)	(4.61, 4.61)
1.0	(1.03, 1.03)	(1.06, 1.05)	(1.02, 1.00)	(0.99, 0.97)	(0.98, 0.95)
1.5	(0.53, 0.48)	(0.55, 0.50)	(0.53, 0.49)	(0.52, 0.47)	(0.51, 0.46)
2.0	(0.37, 0.29)	(0.38, 0.31)	(0.37, 0.30)	(0.37, 0.29)	(0.36, 0.28)
2.5	(0.31, 0.21)	(0.31, 0.22)	(0.31, 0.21)	(0.31, 0.21)	(0.30, 0.20)
3.0	(0.28, 0.17)	(0.28, 0.18)	(0.28, 0.18)	(0.28, 0.17)	(0.27, 0.17)

As the skewness level increases from zero, the in-control performances deteriorate rapidly for all levels of m. For instance, when m=1000 and the underlying distribution is Lognormal, an increase in the skewness from 0.0 to 0.5 causes the exceedance probability $\Pr(CATS_0 \geq \tau)$ to fall from 87.51% to 52.68%, and the $AATS_0$ value to drop from 714.56 to 424.17 (see Table II). This indicates an increased risk of false alarms when the distribution of data transits from symmetric-shaped to skewed-shaped. Besides, as the Phase-I sample size m increases, the ($AATS_0$, $ASDTS_0$) values for all three distributions tend to converge to their corresponding (ATS_0, $SDTS_0$) values in the case of known parameters (m=+∞). While a larger m seems to result in more consistent unconditional performances (i.e., the ($AATS_0$, $ASDTS_0$) values approach the nominal (ATS_0, $SDTS_0$) values), we found that the conditional in-control performance worsens as m increases, especially when the distribution is skewed (i.e., k>0). It is also worth noting that, when m=+∞, only the Weibull distribution with zero skewness produces ATS_0>370.40 with probability one.

TABLE II: $\Pr(CATS_0 \ge \tau)$, $AATS_0$, AND $ASDTS_0$ VALUES OF THE OPTIMAL SPRT CHART WITH ESTIMATED PROCESS PARAMETERS DESIGNED UNDER THE NORMAL MODEL, FOR $m \in \{200, 400, 600, 1000, +\infty\}$, $k \in \{0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}$, AND τ = 370.4, WHEN THE UNDERLYING DISTRIBUTIONS ARE GAMMA, LOGNORMAL, AND WEIBULL

Distribution	Parameter	k	m = 200	m = 400	m = 600	m = 1000	m = +∞
			$\Pr(CATS_0 \ge \tau)$ ($AATS_0$, $ASDTS_0$)	$\Pr(CATS_0 \ge \tau)$ ($AATS_0$, $ASDTS_0$)	$\Pr(CATS_0 \ge \tau)$ ($AATS_0$, $ASDTS_0$)	$\Pr(CATS_0 \ge \tau)$ ($AATS_0$, $ASDTS_0$)	$\Pr(CATS_0 \ge \tau)$ (ATS_0, $SDTS_0$)
Normal	-	0.0	92.77% (5856.56, > 20000)	91.34% (1784.91, 3501.03)	90.24% (1091.10, 1638.63)	87.69% (724.57, 901.03)	100% (370.40, 370.13)
Gamma	α						
	160000	0.0	92.61% (5761.50, > 20000)	91.27% (1762.43, 3484.92)	90.07% (1084.11, 1639.53)	87.46% (717.95, 898.54)	0% (368.36, 368.09)
	16.00000	0.5	83.05% (2471.22, 9237.85)	77.64% (974.13, 1769.69)	69.72% (622.26, 900.15)	53.16% (424.31, 515.70)	0% (229.43, 229.16)
	4.00000	1.0	69.25% (1337.99, 4306.31)	59.50% (624.67, 1079.48)	44.35% (412.39, 573.39)	20.24% (285.70, 339.79)	0% (160.62, 160.36)
	1.77778	1.5	54.91% (845.24, 2665.75)	42.33% (438.75, 741.40)	24.78% (300.90, 411.06)	6.04% (211.18, 248.03)	0% (123.37, 123.11)
	1.00000	2.0	42.85% (596.76, 1748.63)	29.42% (335.50, 564.88)	13.54% (234.77, 318.22)	1.86% (168.29, 196.73)	0% (101.77, 101.51)
	0.64000	2.5	33.37% (461.93, 1427.17)	20.86% (272.85, 463.47)	7.81% (194.62, 263.38)	0.66% (142.16, 166.39)	0% (88.50, 88.23)
	0.44444	3.0	26.90% (388.00, 2560.26)	15.30% (236.03, 398.79)	4.92% (170.75, 231.03)	0.32% (126.41, 147.35)	0% (80.40, 80.13)
Lognormal	σ_{LN}						
	0.00167	0.0	92.60% (5784.73, > 20000)	91.35% (1762.91, 3514.18)	89.99% (1084.30, 1635.58)	87.51% (714.56, 892.05)	0% (368.35, 368.09)
	0.16405	0.5	82.66% (2446.71, 9503.49)	77.62% (974.78, 1777.25)	69.57% (621.96, 895.81)	52.68% (424.17, 511.50)	0% (228.73, 228.47)
	0.31426	1.0	68.64% (1266.82, 4251.24)	58.98% (609.89, 1062.75)	43.60% (407.41, 565.86)	19.48% (282.68, 335.84)	0% (160.34, 160.08)
	0.44349	1.5	53.43% (776.54, 3766.32)	41.40% (430.07, 746.43)	24.04% (297.50, 403.51)	5.83% (211.35, 247.87)	0% (125.51, 125.24)
	0.55138	2.0	41.04% (569.86, 5827.56)	28.91% (337.73, 688.56)	13.76% (239.33, 332.67)	2.23% (174.48, 205.54)	0% (107.39, 107.13)
	0.64088	2.5	32.15% (466.17, 5893.05)	21.24% (290.26, 676.18)	9.14% (206.92, 317.66)	1.34% (154.38, 186.75)	0% (97.71, 97.45)
	0.71557	3.0	26.56% (446.88, > 10000)	16.94% (263.34, 1518.45)	7.08% (191.88, 456.80)	1.20% (144.42, 204.31)	0% (92.44, 92.18)
Weibull	β						
	3.60235	0.0	93.05% (6539.93, > 25000)	91.88% (1859.32, 3741.83)	90.70% (1131.91, 1728.81)	88.76% (748.35, 942.33)	100% (381.74, 381.47)
	2.21560	0.5	83.30% (2695.90, 9535.27)	78.23% (1008.28, 1864.63)	70.57% (644.02, 931.29)	54.71% (436.94, 530.52)	0% (232.78, 232.51)
	1.56391	1.0	69.90% (1411.86, 4456.09)	59.89% (630.25, 1100.92)	44.77% (415.55, 579.67)	20.93% (288.65, 344.64)	0% (160.98, 160.71)
	1.21112	1.5	55.26% (873.10, 2744.46)	42.53% (441.80, 752.08)	24.88% (301.20, 411.58)	6.08% (211.48, 249.77)	0% (122.96, 122.69)
	1.00000	2.0	42.88% (597.04, 1820.98)	29.33% (337.02, 565.39)	13.45% (235.05, 318.21)	1.81% (168.33, 196.67)	0% (101.77, 101.51)
	0.86317	2.5	33.16% (457.03, 2305.04)	20.75% (277.63, 525.44)	7.89% (196.02, 265.84)	0.71% (143.99, 167.57)	0% (89.74, 89.47)
	0.76862	3.0	26.59% (378.92, 2261.98)	15.44% (238.05, 415.72)	5.38% (172.42, 236.30)	0.41% (128.95, 151.13)	0% (83.07, 82.80)

Table III, Table IV, and Table V tabulate the out-of-control metrics, i.e., the ($AATS_1$, $ASDTS_1$) values, of the optimal SPRT chart with GICP-adjusted limits when the underlying distributions are Gamma, Lognormal, and Weibull, respectively. For each skewness k, the results are calculated for $\delta \in \{0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}$. As a numeric example, when m=400, k=2.0, and the underlying distribution is Gamma, the ($AATS_1$, $ASDTS_1$) values at δ=0.5 are equal to (6.93, 9.24), respectively (see Table III).

TABLE III: $AATS_1$ AND $ASDTS_1$ VALUES OF THE OPTIMAL SPRT CHART WITH ESTIMATED PROCESS PARAMETERS DESIGNED UNDER THE NORMAL MODEL, FOR $\delta \in \{0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}$, $k \in \{0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}$, AND $m \in \{200, 400, 600, 1000, +\infty\}$, WHEN THE UNDERLYING DISTRIBUTION IS GAMMA

α	k	δ	m = 200 ($AATS_1$, $ASDTS_1$)	m = 400 ($AATS_1$, $ASDTS_1$)	m = 600 ($AATS_1$, $ASDTS_1$)	m = 1000 ($AATS_1$, $ASDTS_1$)	$m = +\infty$ (ATS_1, $SDTS_1$)
160000	0.0	0.5	(6.64, 9.46)	(5.01, 5.50)	(4.82, 5.14)	(4.72, 4.90)	(4.62, 4.61)
		1.0	(1.03, 1.03)	(1.06, 1.05)	(1.03, 1.00)	(0.99, 0.97)	(0.98, 0.95)
		1.5	(0.53, 0.48)	(0.55, 0.51)	(0.54, 0.49)	(0.52, 0.47)	(0.51, 0.46)
		2.0	(0.37, 0.29)	(0.38, 0.31)	(0.37, 0.30)	(0.37, 0.29)	(0.36, 0.28)
		2.5	(0.31, 0.21)	(0.31, 0.22)	(0.31, 0.21)	(0.30, 0.21)	(0.30, 0.20)
		3.0	(0.28, 0.17)	(0.28, 0.18)	(0.28, 0.18)	(0.28, 0.17)	(0.27, 0.17)
16.00000	0.5	0.5	(7.61, 12.25)	(5.43, 6.18)	(5.20, 5.63)	(5.11, 5.35)	(4.99, 4.98)
		1.0	(1.09, 1.11)	(1.13, 1.13)	(1.09, 1.07)	(1.05, 1.03)	(1.03, 1.01)
		1.5	(0.54, 0.50)	(0.56, 0.52)	(0.55, 0.51)	(0.53, 0.48)	(0.52, 0.47)
		2.0	(0.36, 0.28)	(0.37, 0.30)	(0.37, 0.29)	(0.36, 0.28)	(0.35, 0.27)
		2.5	(0.29, 0.19)	(0.30, 0.20)	(0.30, 0.20)	(0.29, 0.19)	(0.29, 0.19)
		3.0	(0.29, 0.18)	(0.27, 0.16)	(0.28, 0.16)	(0.27, 0.16)	(0.27, 0.16)
4.00000	1.0	0.5	(8.87, 18.93)	(5.86, 6.97)	(5.63, 6.23)	(5.55, 5.89)	(5.39, 5.38)
		1.0	(1.16, 1.19)	(1.20, 1.20)	(1.15, 1.14)	(1.10, 1.09)	(1.09, 1.07)
		1.5	(0.55, 0.51)	(0.58, 0.55)	(0.56, 0.52)	(0.54, 0.49)	(0.53, 0.48)
		2.0	(0.35, 0.27)	(0.36, 0.29)	(0.35, 0.28)	(0.34, 0.26)	(0.33, 0.25)
		2.5	(0.28, 0.17)	(0.28, 0.18)	(0.28, 0.17)	(0.28, 0.17)	(0.27, 0.16)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
1.77778	1.5	0.5	(10.36, 25.34)	(6.38, 7.97)	(6.11, 6.97)	(6.01, 6.47)	(5.81, 5.81)
		1.0	(1.23, 1.29)	(1.27, 1.28)	(1.22, 1.22)	(1.16, 1.15)	(1.14, 1.12)
		1.5	(0.56, 0.53)	(0.60, 0.57)	(0.58, 0.54)	(0.55, 0.50)	(0.53, 0.49)
		2.0	(0.32, 0.24)	(0.34, 0.27)	(0.33, 0.24)	(0.31, 0.22)	(0.31, 0.22)
		2.5	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
1.00000	2.0	0.5	(12.33, 38.99)	(6.93, 9.24)	(6.60, 7.76)	(6.50, 7.14)	(6.27, 6.27)
		1.0	(1.30, 1.39)	(1.34, 1.37)	(1.28, 1.29)	(1.22, 1.22)	(1.20, 1.18)
		1.5	(0.57, 0.55)	(0.63, 0.61)	(0.59, 0.56)	(0.55, 0.51)	(0.54, 0.50)
		2.0	(0.29, 0.19)	(0.30, 0.21)	(0.28, 0.18)	(0.27, 0.16)	(0.26, 0.15)
		2.5	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
0.64000	2.5	0.5	(14.79, 59.06)	(7.56, 10.83)	(7.15, 8.80)	(7.02, 7.87)	(6.77, 6.76)
		1.0	(1.38, 1.52)	(1.42, 1.46)	(1.35, 1.37)	(1.29, 1.29)	(1.26, 1.24)
		1.5	(0.56, 0.58)	(0.65, 0.65)	(0.60, 0.58)	(0.55, 0.52)	(0.54, 0.50)
		2.0	(0.28, 0.17)	(0.27, 0.17)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		2.5	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
0.44444	3.0	0.5	(17.70, 101.46)	(8.27, 13.06)	(7.74, 10.05)	(7.63, 8.80)	(7.14, 7.13)
		1.0	(1.46, 1.66)	(1.53, 1.58)	(1.44, 1.47)	(1.36, 1.37)	(1.33, 1.32)
		1.5	(0.54, 0.62)	(0.65, 0.69)	(0.58, 0.59)	(0.50, 0.49)	(0.51, 0.46)
		2.0	(0.28, 0.17)	(0.27, 0.16)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		2.5	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)

From all the three tables, we notice that when k=0.0, the ($AATS_1$, $ASDTS_1$) values of the SPRT chart under the Gamma, Lognormal, and Weibull distributions are very similar to those obtained under the Normal distribution (see Table I). It is interesting to note, however, that the reported values for the Weibull distribution are slightly lower than those of the other distributions for a small shift size (i.e., δ = 0.5), both in the cases of known and estimated process parameters.

For instance, when m = 400, k=0.0, and δ=0.5, the ($AATS_1$, $ASDTS_1$) values of the SPRT chart under the Gamma (=(5.01, 5.50)) and Lognormal (=(5.03, 5.55)) distributions are quite close to those under the Normal distribution (=(5.02, 5.54)); whereas the Weibull distribution reports slightly better performances (=(4.95, 5.45)) compared to the other three. Overall, when the skewness level increases, the out-of-control performances deteriorate, with the worst case being reported for the smallest m (=200). It is also observed that, when only a small m is available, the out- of-control performances are affected the most when the underlying distribution is Lognormal. This is evident from the cells corresponding to m=200, k=3.0, and δ=0.5 in Table III to Table V. It is noticed that the ($AATS_1$, $ASDTS_1$) values under the Gamma (=(17.70, 101.46)) and Weibull (=(19.56, 228.99)) distributions are not too far from each other, while those under the Lognormal distribution (=(58.69, 3056.37)) are seen to be strikingly large. As m increases, this phenomenon ceases, and the chart's performances stabilize and converge to their corresponding (ATS_1, $SDTS_1$) values in the case of known parameters. However, the deterioration in the out-of-control detection times due to skewed distributions remains a significant concern.

TABLE IV: $AATS_1$ AND $ASDTS_1$ VALUES OF THE OPTIMAL SPRT CHART WITH ESTIMATED PROCESS PARAMETERS DESIGNED UNDER THE NORMAL MODEL, FOR $\delta \in \{0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}$, $k \in \{0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}$, AND $m \in \{200, 400, 600, 1000, +\infty\}$, WHEN THE UNDERLYING DISTRIBUTION IS LOGNORMAL

			m = 200	m = 400	m = 600	m = 1000	m = +∞
σ_{LN}	k	δ	($AATS_1$, $ASDTS_1$)	($AATS_1$, $ASDTS_1$)	($AATS_1$, $ASDTS_1$)	($AATS_1$, $ASDTS_1$)	(ATS_1, $SDTS_1$)
0.00167	0.0	0.5	(6.68, 9.60)	(5.03, 5.55)	(4.81, 5.11)	(4.73, 4.91)	(4.62, 4.61)
		1.0	(1.03, 1.03)	(1.06, 1.05)	(1.03, 1.01)	(0.99, 0.97)	(0.98, 0.95)
		1.5	(0.53, 0.48)	(0.55, 0.50)	(0.54, 0.49)	(0.52, 0.47)	(0.51, 0.46)
		2.0	(0.37, 0.29)	(0.37, 0.31)	(0.37, 0.30)	(0.37, 0.29)	(0.36, 0.28)
		2.5	(0.31, 0.21)	(0.31, 0.22)	(0.31, 0.21)	(0.31, 0.21)	(0.30, 0.20)
		3.0	(0.28, 0.17)	(0.28, 0.18)	(0.28, 0.18)	(0.28, 0.17)	(0.27, 0.17)
0.16405	0.5	0.5	(7.63, 13.31)	(5.43, 6.19)	(5.22, 5.66)	(5.12, 5.36)	(4.99, 4.99)
		1.0	(1.09, 1.10)	(1.13, 1.13)	(1.08, 1.07)	(1.04, 1.02)	(1.03, 1.01)
		1.5	(0.54, 0.49)	(0.56, 0.52)	(0.55, 0.50)	(0.53, 0.48)	(0.52, 0.47)
		2.0	(0.36, 0.28)	(0.37, 0.30)	(0.37, 0.29)	(0.36, 0.28)	(0.35, 0.27)
		2.5	(0.29, 0.19)	(0.30, 0.20)	(0.30, 0.20)	(0.29, 0.19)	(0.29, 0.19)
		3.0	(0.28, 0.16)	(0.27, 0.16)	(0.28, 0.16)	(0.28, 0.16)	(0.27, 0.16)
0.31426	1.0	0.5	(8.95, 19.94)	(5.92, 7.07)	(5.67, 6.31)	(5.56, 5.90)	(5.40, 5.40)
		1.0	(1.15, 1.18)	(1.19, 1.19)	(1.14, 1.13)	(1.09, 1.08)	(1.08, 1.06)
		1.5	(0.54, 0.50)	(0.57, 0.54)	(0.55, 0.51)	(0.53, 0.48)	(0.52, 0.47)
		2.0	(0.35, 0.27)	(0.36, 0.29)	(0.35, 0.28)	(0.34, 0.26)	(0.33, 0.25)
		2.5	(0.28, 0.18)	(0.28, 0.18)	(0.28, 0.18)	(0.28, 0.17)	(0.27, 0.17)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
0.44349	1.5	0.5	(10.91, 40.03)	(6.44, 8.15)	(6.14, 7.05)	(6.02, 6.54)	(5.81, 5.81)
		1.0	(1.20, 1.27)	(1.24, 1.25)	(1.19, 1.19)	(1.13, 1.13)	(1.12, 1.09)
		1.5	(0.54, 0.51)	(0.58, 0.55)	(0.56, 0.51)	(0.53, 0.48)	(0.51, 0.47)
		2.0	(0.33, 0.25)	(0.34, 0.27)	(0.34, 0.25)	(0.32, 0.24)	(0.32, 0.23)
		2.5	(0.27, 0.16)	(0.27, 0.16)	(0.27, 0.16)	(0.27, 0.16)	(0.27, 0.16)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
0.55138	2.0	0.5	(14.83, 296.93)	(7.06, 11.55)	(6.61, 8.17)	(6.49, 7.90)	(6.20, 6.20)
		1.0	(1.25, 3.20)	(1.29, 1.32)	(1.23, 1.24)	(1.17, 1.16)	(1.15, 1.13)
		1.5	(0.53, 0.52)	(0.58, 0.56)	(0.55, 0.51)	(0.52, 0.47)	(0.51, 0.46)
		2.0	(0.31, 0.23)	(0.33, 0.25)	(0.32, 0.23)	(0.30, 0.21)	(0.30, 0.20)
		2.5	(0.27, 0.16)	(0.27, 0.16)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
0.64088	2.5	0.5	(30.62, 5188.91)	(8.29, 275.83)	(7.21, 12.35)	(6.92, 11.44)	(6.56, 6.56)
		1.0	(1.82, 625.89)	(1.33, 1.40)	(1.26, 1.29)	(1.19, 1.19)	(1.17, 1.15)
		1.5	(0.53, 0.53)	(0.58, 0.57)	(0.55, 0.52)	(0.51, 0.47)	(0.50, 0.45)
		2.0	(0.30, 0.21)	(0.31, 0.23)	(0.30, 0.20)	(0.29, 0.18)	(0.28, 0.18)
		2.5	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
0.71557	3.0	0.5	(58.69, 3056.37)	(12.52, 743.25)	(7.84, 24.01)	(7.44, 13.58)	(6.89, 6.89)
		1.0	(7.90, 1973.22)	(1.42, 137.45)	(1.29, 1.34)	(1.21, 1.23)	(1.19, 1.17)
		1.5	(0.52, 5.08)	(0.58, 0.59)	(0.54, 0.52)	(0.50, 0.46)	(0.49, 0.44)
		2.0	(0.29, 0.23)	(0.30, 0.21)	(0.29, 0.19)	(0.28, 0.17)	(0.27, 0.16)
		2.5	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)

V. CONCLUSION

In this paper, we investigate the impact of skewness on the performances of the optimal SPRT chart designed with GICP-adjusted control limits. The study has been conducted by considering three different well-known distributions, i.e., the Gamma, Lognormal, and Weibull distributions. Based on our findings and analyses, we have obtained the following conclusions:

1) An increased skewness level k poses adverse effects on both the conditional and unconditional in-control performances of the SPRT chart with estimated process parameters. In particular, when k increases, the exceedance probability of the SPRT chart drops rapidly, hitting percentages as low as 0.32% when k=3.0. This is an alarming sign, since practitioners are exposed to higher risks of false alarms under skewed datasets. Besides, the $AATS_0$ values are found to decrease as k increases, indicating worse in-control performance in the unconditional sense.

2) The out-of-control performances of the SPRT chart with estimated process parameters are found to deteriorate as the skewness level k increases. This situation is especially severe when the underlying distribution is Lognormal and the Phase-I sample size is small.

3) It is found that a larger m tends to lead to poorer in-control performance, but a contrastingly better out-of-control performance. This phenomenon aligns with the commonly observed “trade-off” between in-control and out-of-control performances, as found in scenarios involving the adjustment of p and/or ε to control the rigidity of the GICP constraint.

Regarding the appropriate choice of m in practice, we recommend practitioners to carefully assess i) the degree of skewness of the data, ii) the relevant in-control policy set by the factory, and iii) the mean shift size to be targeted in the course of SPM, before proceeding with any sort of decision-making. For instance, if a practitioner expects a small mean shift (e.g., δ=0.5) to take place, then it may not be wise to pursue a small m merely for the purpose of preserving good in-control performance, since the out-of-control performance would be considerably compromised for such a small δ.

TABLE V: $AATS_1$ AND $ASDTS_1$ VALUES OF THE OPTIMAL SPRT CHART WITH ESTIMATED PROCESS PARAMETERS DESIGNED UNDER THE NORMAL MODEL, FOR $\delta \in \{0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}$, $k \in \{0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}$, AND $m \in \{200, 400, 600, 1000, +\infty\}$, WHEN THE UNDERLYING DISTRIBUTION IS WEIBULL

			m = 200	m = 400	m = 600	m = 1000	m = +∞
β	k	δ	($AATS_1$, $ASDTS_1$)	($AATS_1$, $ASDTS_1$)	($AATS_1$, $ASDTS_1$)	($AATS_1$, $ASDTS_1$)	(ATS_1, $SDTS_1$)
3.60235	0.0	0.5	(6.54, 9.09)	(4.95, 5.45)	(4.75, 5.04)	(4.67, 4.84)	(4.55, 4.55)
		1.0	(1.03, 1.03)	(1.06, 1.05)	(1.02, 1.00)	(0.99, 0.96)	(0.98, 0.95)
		1.5	(0.53, 0.49)	(0.55, 0.51)	(0.54, 0.49)	(0.52, 0.48)	(0.51, 0.47)
		2.0	(0.37, 0.30)	(0.38, 0.31)	(0.38, 0.31)	(0.37, 0.30)	(0.36, 0.29)
		2.5	(0.31, 0.21)	(0.31, 0.22)	(0.31, 0.22)	(0.31, 0.21)	(0.30, 0.21)
		3.0	(0.28, 0.17)	(0.28, 0.17)	(0.28, 0.17)	(0.28, 0.17)	(0.27, 0.17)
2.21560	0.5	0.5	(7.47, 12.12)	(5.35, 6.07)	(5.14, 5.56)	(5.06, 5.30)	(4.93, 4.92)
		1.0	(1.10, 1.11)	(1.12, 1.12)	(1.09, 1.08)	(1.05, 1.03)	(1.04, 1.01)
		1.5	(0.55, 0.51)	(0.58, 0.54)	(0.56, 0.52)	(0.54, 0.49)	(0.53, 0.48)
		2.0	(0.37, 0.29)	(0.38, 0.31)	(0.37, 0.30)	(0.36, 0.29)	(0.35, 0.28)
		2.5	(0.29, 0.19)	(0.29, 0.20)	(0.29, 0.19)	(0.29, 0.19)	(0.28, 0.18)
		3.0	(0.27, 0.16)	(0.27, 0.16)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
1.56391	1.0	0.5	(8.70, 16.78)	(5.80, 6.83)	(5.58, 6.13)	(5.51, 5.84)	(5.35, 5.35)
		1.0	(1.17, 1.20)	(1.20, 1.20)	(1.16, 1.15)	(1.11, 1.10)	(1.10, 1.08)
		1.5	(0.56, 0.52)	(0.60, 0.56)	(0.58, 0.53)	(0.55, 0.50)	(0.54, 0.49)
		2.0	(0.35, 0.27)	(0.37, 0.30)	(0.36, 0.28)	(0.34, 0.26)	(0.34, 0.26)
		2.5	(0.27, 0.16)	(0.27, 0.16)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
1.21112	1.5	0.5	(10.19, 23.56)	(6.34, 7.89)	(6.07, 6.89)	(5.99, 6.41)	(5.81, 5.80)
		1.0	(1.24, 1.29)	(1.27, 1.28)	(1.22, 1.22)	(1.17, 1.16)	(1.15, 1.13)
		1.5	(0.57, 0.54)	(0.61, 0.58)	(0.59, 0.55)	(0.55, 0.51)	(0.54, 0.50)
		2.0	(0.32, 0.23)	(0.34, 0.26)	(0.32, 0.24)	(0.31, 0.21)	(0.30, 0.20)
		2.5	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
1.00000	2.0	0.5	(12.33, 41.18)	(6.94, 9.42)	(6.60, 7.80)	(6.49, 7.13)	(6.27, 6.27)
		1.0	(1.30, 1.40)	(1.34, 1.37)	(1.28, 1.29)	(1.22, 1.21)	(1.20, 1.18)
		1.5	(0.57, 0.55)	(0.63, 0.61)	(0.59, 0.56)	(0.55, 0.51)	(0.54, 0.50)
		2.0	(0.29, 0.19)	(0.30, 0.21)	(0.28, 0.18)	(0.27, 0.16)	(0.26, 0.15)
		2.5	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
0.86317	2.5	0.5	(15.22, 76.83)	(7.60, 11.25)	(7.15, 8.82)	(7.04, 7.94)	(6.73, 6.73)
		1.0	(1.35, 1.50)	(1.40, 1.44)	(1.33, 1.35)	(1.27, 1.27)	(1.24, 1.22)
		1.5	(0.55, 0.57)	(0.64, 0.63)	(0.59, 0.57)	(0.54, 0.51)	(0.53, 0.48)
		2.0	(0.28, 0.18)	(0.28, 0.18)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		2.5	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
0.76862	3.0	0.5	(19.56, 228.99)	(8.35, 15.32)	(7.70, 10.38)	(7.55, 8.73)	(7.17, 7.17)
		1.0	(1.42, 2.44)	(1.47, 1.54)	(1.38, 1.42)	(1.31, 1.32)	(1.28, 1.26)
		1.5	(0.53, 0.59)	(0.63, 0.65)	(0.57, 0.56)	(0.51, 0.49)	(0.51, 0.46)
		2.0	(0.28, 0.18)	(0.27, 0.17)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		2.5	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)
		3.0	(0.27, 0.16)	(0.27, 0.15)	(0.27, 0.16)	(0.27, 0.16)	(0.26, 0.15)

In future research, it would be valuable to modify the GICP method so that the designed SPRT control chart is more robust towards skewed distributions. Other possible future works include devising a non-parametric SPRT chart with estimated process parameters, or designing specific parametric SPRT charts for the Gamma, Lognormal, and Weibull distributions, etc.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

AUTHOR CONTRIBUTIONS

J.W. Teoh, W.L. Teoh, Z.L. Chong, M.H. Lee, and S.Y. Teh contributed to the theoretical development of the project, computer programming, and analyses of the results. W.L. Teoh provided the conceptualization of the research. J.W. Teoh wrote the paper. W.L. Teoh, Z.L. Chong, M.H. Lee, and S.Y. Teh reviewed, edited, and improved the write-up of the paper. All authors had approved the final version.

ACKNOWLEDGMENT

This work was supported by the Ministry of Higher Education (MOHE) Malaysia and Heriot-Watt University

Malaysia under the Fundamental Research Grant Scheme (FRGS), no. FRGS/1/2021/STG06/HWUM/02/1.

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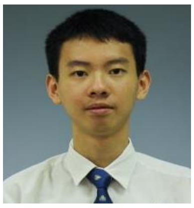

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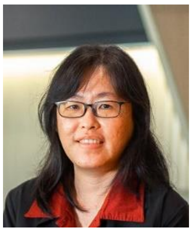

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